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Residual measures in locally compact spaces

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Abstract

A σ-finite diffused Borel measure in a topological space is called *residual* if each nowhere dense set has measure zero. If the measure is also fully supported, then it is called *normal*. Results on the influence of Martin's Axiom and the Continuum Hypothesis on the existence of residual and normal measures in locally compact spaces are obtained. A connection with L-spaces is established. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

A σ-finite diffused Borel measure in a topological space is called *residual* if each nowhere dense set has measure zero. The Lebesgue measure in the real line that is provided with the density topology provides a nice example of a residual measure: The topology and measure are nicely related—a set is nowhere dense iff it is meager iff it is closed and discrete iff it is of outer measure zero. Other nice examples are provided by Stone spaces of measurable algebras: Spaces that are extremally disconnected and compact and are inhabited by a finite measure which zero sets are exactly those that are nowhere dense. In such a space, every measurable essentially bounded function is continuous modulo measure zero. (More details are provided in Section 2.)

Survey articles [6–8], where residual measures are called *hyperdiffuse*, provide valuable information on the topic, as well as extensive bibliography. Other sources are [19, 2, 24]. A brief review of residual measures can be found also in [11]. The famous classical

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work [3] of Dixmier deals with a related notion of *normal measure*, cf. Definition 2.1 below.

There are topological properties that are hostile to residual measures. For instance, in [22], Edwin Szpilrajn proved that a separable dense-in-itself metric space has no finite nontrivial residual Borel measures. In [18], Szpilrajn's result was generalized to an arbitrary metric space without isolated points of topological weight less than the first real-valued measurable cardinal. The cardinal restriction was shown to be superfluous in [6]. Among other results let us mention, e.g., that of [24]—a locally separable Hausdorff space without isolated points does not admit a finite nontrivial residual Borel measure, and [17, Lemma 6.11]—a nondiscrete compact zero-dimensional topological group does not admit a nontrivial finite residual measure.

The present paper continues [24] in that it examines if residual and normal measures exist in some kinds of spaces. The problem (raised in [24]) if a first countable, locally compact space admits a finite residual measure is examined. Namely and mainly we show that if X is a locally compact first countable dense-in-itself space, then, assuming the Martin's Axiom and the negation of the Continuum Hypothesis, there are no residual measures in X. We also show that if the Continuum Hypothesis is assumed, then the conclusion fails, so the question whether or not there are residual measures in first countable compact spaces is independent of the Zermelo–Fraenkel set theory including the Axiom of Choice. This is done in Section 3.

In Section 4 we establish a relation between residual measures in locally compact spaces on one hand and L-spaces on the other hand: Assuming the Continuum Hypothesis, if a locally compact space without isolated points which character does not exceed the continuum admits a finite residual measure, then it contains a subspace that is hereditarily Lindelöf but not hereditarily separable.

Section 2 contains preliminary material and basic definitions and facts related to residual measures. Some facts on so-called Jordan measures (a notion that is weaker than normal and stronger than residual measure, see Definition 2.1, Fact 2.2, Example 2.3 and Remark 2.4) are established and some examples are provided.

2. Residual, Jordan and normal measures

To avoid complications and trivialities, all topological spaces we work with are assumed to be Hausdorff, so the term "space" refers to a Hausdorff topological space. A space is dense-in-itself if it has no isolated points. If X is a space, $\mathbb{B}(X)$ denotes the σ-algebra of all Borel sets. A *Borel measure* μ *in* X (or just a *measure*, if there is no danger of confusion) is a σ-additive mapping $\mu : \mathbb{B}(X) \to [0, \infty]$.

All sets that differ from a Borel set by a negligible set, i.e., by a set contained in a Borel set of measure zero, are considered measurable. A measure μ is *outer regular* if each Borel set is contained in a G_δ-set of the same measure. If μ is finite, then it is obviously outer regular iff each Borel set contains an F_σ-set of the same measure, and in that case μ is said to be *regular*.

For $A \subseteq X$ we denote $\mu^\sharp(A) = \inf\{\mu(B)\colon A \subseteq B \in \mathbb{B}(X)\}$ the outer measure of A. If $Y \subseteq X$ is a subspace, and μ and ν are measures in X and Y, respectively, we define $\mu_Y(B) = \mu^\sharp(B)$ for $B \in \mathbb{B}(Y)$ and ${}_X\nu(B) = \nu(B \cap Y)$ for $B \in \mathbb{B}(X)$. These definitions indeed define measures in Y and X, respectively, cf. [12, Section 3].

A measure μ in X is *trivial* if $\mu X = 0$. It is *locally trivial* if there is an open cover of X by negligible sets. The *support* of μ, denoted by $\operatorname{spt}\mu$, is the set of the points each neighborhood of which has positive measure. Equivalently, $\operatorname{spt}\mu$ is the intersection of all closed sets of full measure (and is thus closed). If each nonempty open set of X has positive measure, μ is called *strictly positive*. Clearly μ is strictly positive iff $\operatorname{spt}\mu = X$. A measure μ is *fully supported* if $\mu(X \setminus \operatorname{spt}\mu) = 0$, i.e., if the union of all open negligible sets is negligible. Note that μ is locally trivial iff $\operatorname{spt}\mu = \emptyset$. A measure μ is *τ-additive* if $\mu(\bigcap \mathcal{F}) = \inf_{F \in \mathcal{F}} \mu F$ for each downward directed family $\mathcal{F}$ of closed sets.

A measure μ is *finite* if $\mu X < \infty$, and *σ-finite* if X admits a countable cover by measurable sets of finite measure. It is *diffused* if it vanishes on singletons.

We refer to the survey article [12] for measure-theoretic notions that we forgot to define.

The symbol $\mathbb{R}$ denotes the set of reals. The symbols ω and ω_1 denote, respectively, the first infinite cardinal = the set of natural numbers including zero and the first uncountable cardinal. The cardinal of continuum is denoted by $\mathfrak{c}$. A cardinality of a set A is denoted by $|A|$. If X is a space and $A \subseteq X$, then $\overline{A}$ and $\operatorname{int} A$ denote, respectively, the closure and interior of A in X. Recall that A is *meager* if it is a countable union of nowhere dense sets and that X is *Baire* if no nonempty open set in X is meager. The term "X is ccc" refers to "X satisfies the countable chain condition".

Definition 2.1. *A σ-finite measure μ in a space X is called*

(i) residual *if $\mu A = 0$ whenever $A \subseteq X$ is meager,*
(ii) Jordan *if $\mu \overline{A} = 0$ whenever $A \subseteq X$ is meager,*
(iii) normal *if it is simultaneously residual and fully supported.*

As far as I know, the term *residual* is due to Armstrong and Prikry [2] and *normal* to Dixmier [3]. The definition of Jordan measure was taken from [6, p. 120].

The notion of a σ-finite measure that is simultaneously Jordan and fully supported is not given a special consideration. Fact 2.2(ii) below explains why.

The notion of normal measure is closely related to that of category measure introduced by Oxtoby, cf. [19]: A finite measure μ in a space X is called a *category measure* if the σ-ideals of meager sets and of μ-negligible sets coincide. Obviously, if μ is a normal measure, then $\mu_{\operatorname{spt}\mu}$ is a strictly positive residual measure in $\operatorname{spt}\mu$, and provided X is a Baire space, a finite measure in X is strictly positive and residual iff it is a category measure.

Recall that a space is *quasiregular* if each nonempty open set contains a closure of some nonempty open set. It is easy to verify that quasiregularity is hereditary with respect to dense or open subspaces. The following are basic properties and relationships of notions from Definition 2.1. (i) is from [1], (ii) is [6, Proposition 2.14] and (iii) is obvious.

Fact 2.2. *Let μ be a finite measure in a space X.*

(i) *If μ is normal, then μ is τ-additive. If, moreover, X is quasiregular, then μ is regular.*

(ii) *If μ is residual and regular, then μ is Jordan. In particular, if μ is normal in a quasiregular space, then μ is Jordan.*

(iii) *If μ is Jordan, then μ is residual.*

Example 2.3. A finite measure in a Hausdorff space that is strictly positive and Jordan, and yet it is not regular. (The underlying space cannot be quasiregular, as explained in Remark 2.4 below.)

Let $X = [0, 1]$ and λ be the Lebesgue measure in X. Let $D \subseteq X$ be a set satisfying $\lambda^\sharp(D) = \lambda^\sharp(X \setminus D) = 1$. Denote by σ the density topology in X. Define τ to be the coarsest topology that is finer than σ and makes the set D open. As τ is finer than σ, it is Hausdorff. It is a matter of routine to show that τ has the following properties.

(i) D is open and dense,

(ii) $F \subseteq X$ is nowhere dense iff $F \cap D$ is nowhere dense in the density topology,

(iii) if $F \subseteq D$ is closed, then F is nowhere dense.

For each Borel set B (with respect to τ) put $\mu B = \lambda^\sharp(B \cap D)$. It is easy to show that μ is a Borel measure in $\langle X, \tau\rangle$. Obviously μ is diffused and finite, and since $\lambda^\sharp D = 1$, it is nontrivial and strictly positive. As λ is Jordan in σ, (ii) implies that μ is Jordan. By (i) and (iii), $\mu D = 1$ and yet $\mu F = 0$ for each closed $F \subseteq D$, whence μ is not regular.

Remark 2.4. We discuss the influence of real-valued measurable cardinals on the relations described in Fact 2.2. A cardinal κ is called *real-valued measurable* if the discrete space κ admits a diffused probability Borel measure that is κ-additive, i.e., a union of less than κ many negligible sets remains negligible. We denote by $\mathfrak{r}$ the least real-valued measurable cardinal.

Under the absence of real-valued measurable cardinals

- all finite residual measures are normal [6, Theorem 2.6],
- hence Fact 2.2(iii) and (i) yield that if there are no real-valued measurable cardinals, then each finite Jordan measure in a quasiregular space is regular.

If there is a real-valued measurable cardinal, then

- There are finite regular Jordan measures in regular spaces that are not normal: Consider a real-valued measurable cardinal κ and provide it with a discrete topology and a measure λ witnessing to real-valued measurability of κ. Then λ is locally trivial, regular and Jordan.
- There are finite Jordan measures in regular spaces that are not regular: Consider the preceding example and let $X = \kappa \cup \{\kappa\}$ be the one-point compactification of κ. Then $\mu = {}_X\lambda$ is Jordan by Claim 2.9(ii) below. Since each G_δ-set containing κ is a complement of a countable set, it has full measure. It follows that μ is not regular. Using the idea of [24, Example 3.4], both examples can be modified to have no isolated points.

- On the other hand, I do not know an example of a Hausdorff (or even regular) space admitting a finite residual measure that is not Jordan.

Normal measures in quasiregular spaces behave particularly well. The following is partially gathered from [1, 19] and [6, Theorem 2.15], partially obvious.

Fact 2.5. *Let X be a quasiregular space, μ a finite diffused normal measure in X and $G \subseteq \operatorname{spt}\mu$ a nonempty set that is open in* $\operatorname{spt}\mu$.

(i) *A subset $F \subseteq G$ is meager iff it is nowhere dense iff $\mu F = 0$. Thus G is a Baire space, $\mu G > 0$ and μ_G is a strictly positive residual measure in G.*

(ii) *G is not separable.*

(iii) *G is ccc.*

Jordan measures are characterized by the behavior of measurable functions. The equivalence (i) $\Leftrightarrow$ (ii) below is proved, e.g., in [6, p. 119].

Proposition 2.6. *For a finite measure μ in a space X, the following are equivalent.*

(i) *μ is Jordan.*

(ii) *$\mu(E) = \mu(\overline{E}) = \mu(\operatorname{int} E)$ for each measurable set $E \subseteq X$.*

(iii) *Each measurable function $f : X \to \mathbb{R}$ is continuous almost everywhere.*

(iv) *Each measurable function $f : X \to \mathbb{R}$ is continuous on an open set of full measure.*

Proof. (ii) $\Rightarrow$ (iv) Let $f : X \to \mathbb{R}$ be measurable. Denote by $\mathbb{Q}$ the set of rationals. For $r, s \in \mathbb{Q}$, put $E_{r,s} = \overline{f^{-1}(r,s)} \setminus \operatorname{int} f^{-1}(r,s)$, $F = \bigcup_{r,s\in\mathbb{Q}} E_{r,s}$ and $H = \overline{F}$. Since $f^{-1}(r,s)$ is measurable, it follows from (ii) that $E_{r,s}$ has measure zero. Hence, again by (ii), $\mu H = 0$: for $\mathbb{Q}$ is countable. Let $x \in X \setminus H$ and $r, s \in \mathbb{Q}$ such that $r < fx < s$. Then $x \in \operatorname{int} f^{-1}(r,s)$, for otherwise $x \in f^{-1}(r,s) \setminus \operatorname{int} f^{-1}(r,s) \subseteq E_{r,s} \subseteq H$. We have shown that f is continuous at each point of $X \setminus H$, an open set of full measure.

(iii) $\Rightarrow$ (i) Let $E \subseteq X$ be a measurable set and χ_E its characteristic function. For $i \in \{0, 1\}$, let A_i be the set of those points that have a neighborhood U such that $\chi_E(y) = i$ for each $y \in U$. Let C be the set of points of continuity of χ_E. As A_i's are open and χ_E is $\{0, 1\}$-valued, $C \subseteq A_0 \cup A_1$. It is obvious that $A_0 \subseteq E$ and $A_1 \subseteq X \setminus E$.

If E is nowhere dense, then A_0 is clearly empty. Therefore $C \subseteq A_1 \subseteq X \setminus E$ and since C has full measure by (iii), it follows that $\mu E \leqslant \mu(X \setminus C) = 0$. So μ is residual. Hence if E is meager, then

$$\mu\overline{E} \leqslant \mu(X \setminus A_1) = \mu\bigl(X \setminus (A_0 \cup A_1)\bigr) \leqslant \mu(X \setminus C) = 0. \qquad \square$$

Combining Facts 2.2 and 2.5 and Proposition 2.6 yields the following particular case related to the *Dixmier's Theorem* (cf. the text following Corollary 2.7).

Corollary 2.7. *Let X be a quasiregular space and μ a finite strictly positive residual measure in X. For a function $f : X \to \mathbb{R}$, the following are equivalent.*

(i) *f is μ-measurable,*
(ii) *f is continuous at a dense set,*
(iii) *f is continuous at a dense open set,*
(iv) *f is continuous almost everywhere.*

It is easy to check that if μ and ν are two finite diffused measures and μ is residual and fully supported, then ν is absolutely continuous with respect to μ if and only if ν is residual and fully supported and $\operatorname{spt}\nu \subseteq \operatorname{spt}\mu$. So the Radon–Nikodým Theorem and Corollary 2.7 yield that if μ and ν are two finite, diffused, strictly positive, residual measures in a quasiregular space, then $L^p(\mu) \cong L^p(\nu)$ for each $p \in [1,\infty]$. So L^p's depend only on the support. Another corollary to 2.7, essentially the *Dixmier's Theorem*, is that X is extremally disconnected if and only if $L^\infty(\mu) \cong C^*(X)$ (= bounded continuous functions on X), and in that case $L^1(\mu)$ is obviously a predual of $C^*(X)$. If X is moreover compact, then by virtue of Riesz Theorem, Radon–Nikodým Theorem and [7, Proposition 2.4] $L^1(\mu)$ embeds into the Banach space of all finite signed Radon measures in X as a complemented subspace.

Speaking of residual measures one cannot but mention a notion of a hyperstonian space. Let Σ be a σ-algebra of subsets of a set A and let $\lambda:\Sigma \to [0,\infty)$ be a finite atomless measure on Σ. Denote by $\mathcal{I}_\lambda$ the σ-ideal of λ-negligible sets and by $\mathcal{B}_\lambda$ the quotient algebra $\Sigma\big/\mathcal{I}_\lambda$. Then $\mathcal{B}_\lambda$ is a complete Boolean algebra. Its Stone space, $\mathcal{S}_\lambda$, is called a *hyperstonian space* of λ. The Stonian isomorphism of $\mathrm{RO}(\mathcal{S}_\lambda)$, the Boolean algebra of regular open sets of $\mathcal{S}_\lambda$, and $\mathcal{B}_\lambda$, maps λ onto the regular open sets of $\mathcal{S}_\lambda$. Since $\mathcal{S}_\lambda$ is extremally disconnected, it follows that the new measure has a unique extension, $\hat{\lambda}$, on Borel sets of $\mathcal{S}_\lambda$, and it turns out that $\hat{\lambda}$ is a strictly positive residual measure. More information on hyperstonian spaces is provided, e.g., in [2].

Hyperstonian spaces provide useful examples of spaces admitting normal measures. For instance, if λ is the Lebesgue measure on the unit interval, then the topological weight of $\mathcal{S}_\lambda$ is $\mathfrak{c}$. Hence $\mathcal{S}_\lambda$ embeds into the Tychonoff cube $[0,1]^{\mathfrak{c}}$. This yields a claim that will be useful.

Claim 2.8. *The Tychonoff cube $[0,1]^{\mathfrak{c}}$ contains a dense-in-itself subspace X such that there is a finite normal diffused measure in X.*

Extensions and restrictions of residual measures:

Claim 2.9. *Let X be a space and $Y \subseteq X$ a dense subspace. Let μ and ν be finite measures in X and Y, respectively.*

(i) *If μ is residual, then so is μ_Y. If μ is Jordan or normal, respectively, then so is μ_Y and $\mu_Y(Y) = \mu(X)$. Thus if μ is residual or normal, respectively, and nontrivial, then so is μ_Y.*
(ii) *If ν is residual or Jordan or normal, respectively, then so is ${}_X\nu$.*

Proof. If F is a nowhere dense set in Y, then $\overline{F}^X$ is nowhere dense in X, and if H is nowhere dense in X, then $H \cap Y$ is nowhere dense in Y. The proof consists of a straightforward application of these obvious facts. Proving normality of the measures one has to employ also Fact 2.2(i). □

The goal of the paper is to show that in some types of spaces there are no residual or normal measures, respectively. Here are the relevant definitions, 2.10(i) is taken from [24, Definition 1.3(i)].

Definition 2.10. A Hausdorff space X is called

(i) schismatic if for each σ-finite diffused measure μ in X there is a meager set F such that $\mu(X \setminus F) = 0$,
(ii) pre-schismatic if for each σ-finite fully supported diffused measure μ in X there is a meager set F such that $\mu(X \setminus F) = 0$,
(iii) weakly schismatic if for each σ-finite diffused measure μ in X there is a meager set F such that $\mu(X \setminus \overline{F}) = 0$.

The following is easy to prove, Fact 2.11(i) is [24, Claim 1.6].

Fact 2.11. *A space X is*

(i) *schismatic if and only if there are no nontrivial finite diffused residual measures in X,*
(ii) *pre-schismatic if and only if there are no nontrivial finite diffused normal measures in X,*
(iii) *weakly schismatic if and only if there are no nontrivial finite diffused Jordan measures in X.*

Claim 2.9 yields a simple and useful fact.

Claim 2.12. *Let X be a space and $Y \subseteq X$ a dense subspace.*

(i) *If X is schismatic, then so is Y.*
(ii) *X is weakly schismatic if and only if Y is weakly schismatic.*
(iii) *X is pre-schismatic if and only if Y is pre-schismatic.*

It is also clear from the definitions and Fact 2.2 that a schismatic space is weakly schismatic and that a weakly schismatic quasiregular space is pre-schismatic. And obviously if a space X is perfect (i.e., closed sets are G_δ), then each finite measure in X is regular and therefore X is schismatic iff it is weakly schismatic.

Let us clarify why we do not take in account non-diffused measures. A Dirac measure is residual iff it sits on an isolated point, so otherwise any space with an isolated point would be disqualified.

Take, however, a closer look at isolated points. If X is a space and I is the set of its isolated points, then, as I is an open set, X is schismatic iff both I and $X \setminus \overline{I}$ are schismatic, because the remainder is a nowhere dense set. The space $X \setminus \overline{I}$ has no longer isolated points and I is discrete. And a discrete space I is obviously schismatic iff $|I|$ is less than the first real-valued measurable cardinal (see Section 3 for the definition). So the only interesting case is that of X without isolated points. Therefore we will not bother with isolated points at all. (Details are worked out in [24].)

3. Locally compact spaces

In this section we attempt to prove that in first countable locally compact spaces there are no residual measures. It turns out that within ZFC, the Zermelo–Fraenkel set theory including the Axiom of Choice, it is impossible: In [15], Kunen constructed, under the assumption of the Continuum Hypothesis, a compact, first countable space admitting a finite, strictly positive residual measure. The space is zero-dimensional and hereditarily Lindelöff. We will refer to this example as *Kunen's space*. Our point is that, on the contrary, Martin's Axiom plus the negation of Continuum Hypothesis kill residual measures in first countable compact spaces. The main result is given in Theorems 3.5 and 3.8.

We need the machineries of Martin's Axiom, topological cardinal functions, and also that of measurable cardinals. We use the usual ordinal and cardinal notation. If X is a topological space, then $d(X)$ and $w(X)$ denote, respectively, the density and the weight of X. If $x \in X$, then $\chi(x, X)$ denotes the character of X at x. The character of X is defined by $\chi(X) = \min\{\kappa\colon \chi(x, X) \leqslant \kappa \text{ for each } x \in X\}$, and the *augmented character* is defined by $\widehat{\chi}(X) = \min\{\kappa\colon \chi(x, X) < \kappa \text{ for each } x \in X\}$. A family of nonempty open sets $\mathcal{G}$ is a *π-base at x* if each neighborhood of x contains some element of $\mathcal{G}$. A *π-character at x* is defined by $\pi\chi(x, X) = \min\{|\mathcal{G}|\colon \mathcal{G} \text{ is a } \pi\text{-base at } x\}$.

The notion of a real-valued measurable cardinal is recalled in Remark 2.4. Recall that the least real-valued measurable cardinal is denoted by $\mathfrak{r}$. All properties of real-valued measurable cardinals and $\mathfrak{r}$ that we appeal to can be found in [10] or [4].

Martin's Axiom and the countable Martin's Axiom are set-theoretic statements that are known to be equivalent to the following so called *topological versions of Martin's Axiom*: If κ is a cardinal, $\mathrm{MA}(\kappa)$ is the statement

No ccc compact space is a union of κ many nowhere dense sets.

The cardinal $\mathfrak{m}$ is defined as the first cardinal κ for which $\mathrm{MA}(\kappa)$ fails. So $\mathrm{MA}(\kappa)$ iff $\kappa < \mathfrak{m}$. MA is the statement $\mathfrak{m} = \mathfrak{c}$ and $\mathrm{MA} + \neg\mathrm{CH}$ is the statement $\mathfrak{m} = \mathfrak{c} > \omega_1$.

The countable Martin's Axiom, $\mathrm{MA}_{\mathrm{countable}}(\kappa)$, is the statement

No metric compact space is a union of κ many nowhere dense sets

and $\mathfrak{m}_{\mathrm{countable}}$ is the first cardinal κ for which $\mathrm{MA}_{\mathrm{countable}}(\kappa)$ fails. So $\mathrm{MA}(\kappa)$ implies $\mathrm{MA}_{\mathrm{countable}}(\kappa)$ and $\mathfrak{m} \leqslant \mathfrak{m}_{\mathrm{countable}} \leqslant \mathfrak{c}$. The following theorem of [10] relates $\mathfrak{m}_{\mathrm{countable}}$ and $\mathfrak{r}$.

Lemma 3.1. *If $\mathfrak{m}_{\mathrm{countable}} > \omega_1$, then $\mathfrak{c} < \mathfrak{r}$.*

Lemma 3.2 (Assume $\mathfrak{c} < \mathfrak{r}$). *Let X be a locally compact space such that $\chi(X) < \mathfrak{r}$ and Y a dense-in-itself subspace of X. Let μ be a nontrivial finite residual measure in Y. Then*

(i) *μ is Jordan,*

(ii) *there exists a finite nontrivial normal measure in Y.*

Proof. According to Claim 2.9, we may assume that Y is closed. Since a closed subspace of a locally compact space is locally compact and $\chi(Y) \leqslant \chi(X)$, it is actually enough to prove the lemma for X itself in place of Y.

Consider the family $\mathcal{U} = \{U \subseteq X\colon\ U \text{ open}, \overline{U} \text{ compact}, \mu U = 0\}$. Set $\Omega = \bigcup \mathcal{U}$, $\nu = \mu_{\operatorname{spt}\mu}$ and $\lambda = \mu_\Omega$. The spaces $\operatorname{spt}\mu$ and Ω are locally compact and ν is normal in $\operatorname{spt}\mu$ and λ is residual and locally trivial in Ω. So in order to prove (i) it obviously suffices to show that λ is Jordan. Assume that λ is nontrivial, otherwise there is nothing to prove. Let $\{V_\alpha\colon\ \alpha \in I\}$ be a maximal disjoint subfamily of $\mathcal{U}$. Then $G = \bigcup_{\alpha\in I} V_\alpha$ is an open dense subset of Ω and therefore $\mu G = \mu\Omega$.

For $D \subseteq I$, set $\phi(D) = \mu(\bigcup_{\alpha\in D} V_\alpha)$. Then ϕ is a diffused measure in the discrete space I witnessing to $|I| \geqslant \mathfrak{r}$. By virtue of the *Ulam's Dichotomy*, ϕ is a purely atomic measure and is therefore a countable sum of two-valued measures. It is easy to check that being Jordan is a countably additive property, so we may assume, without loss of generality, that ϕ is two-valued. It is well known that each finite measure in a discrete space is $\mathfrak{r}$-additive (i.e., a union of $< \mathfrak{r}$ negligible sets is negligible), so ϕ is $\mathfrak{r}$-additive.

For $\alpha, \beta \in I$, write $\alpha \cong \beta$ if V_α is homeomorphic to V_β. The relation $\cong$ is obviously an equivalence on I. Since $w(K) \leqslant 2^{\chi(K)}$ holds for each compact K (see [13, 2.21]) and $\overline{V}_\alpha$'s are compact, for each $\alpha \in I$ we have $w(V_\alpha) \leqslant 2^{\chi(X)}$, hence each V_α embeds into $[0,1]^{2^{\chi(X)}}$. Therefore $\cong$ has at most $2^{2^{\chi(X)}}$ classes of equivalence. As $\mathfrak{c} < \mathfrak{r}$, the cardinal $\mathfrak{r}$ is strongly inaccessible [10]. Thus the number of equivalence classes of $\cong$ is less than $\mathfrak{r}$. As ϕ is $\mathfrak{r}$-additive and two-valued, it follows that there is a space W and a set $D \subseteq I$ such that $\phi(D) = \phi(I)$ and V_α is homeomorphic to W for each $\alpha \in D$. In other words, the open set $G_0 = \bigcup_{\alpha\in D} V_\alpha$ of X is homeomorphic to $W \times D$ and $\mu(G_0) = \phi(D)$. For each Borel set $E \in \mathbb{B}(W)$ define $\psi(E) = \phi(E \times D)$. Obviously, ψ is a finite Borel measure in W and if E is nowhere dense in W, then, as X has no isolated points, $E \times D$ is nowhere dense in X. Therefore ψ is residual. We show that ψ is fully supported. If not, then W contains a disjoint family $\mathcal{G}$ of open ψ-negligible sets such that $\psi(\bigcup \mathcal{G}) > 0$, which in turn implies that $|\mathcal{G}| \geqslant \mathfrak{r}$. But, as shown above, $w(W) \leqslant 2^{\chi(X)} < \mathfrak{r}$. So ψ is a normal measure in W.

Identify G_0 with $W \times D$. Let $E \in \mathbb{B}(W \times D)$. Since $|\mathbb{B}(W)| < \mathfrak{r}$, there is a set $J \subseteq D$ and a Borel set $B \in \mathbb{B}(W)$ such that $\phi J = \phi D$ and $E \cap (W \times \{j\}) = B$ for each $j \in J$. Therefore $\lambda(E) = \lambda(B \times J) = \psi(B)$. Obviously $B \times J \subseteq E$. Since ψ is normal, it follows from Fact 2.2(i) that there is an F_σ-set $F \subseteq B$ such that $\psi F = \psi B$. The set $F \times J$ is clearly F_σ in $W \times D$ and we conclude that λ restricted to G_0 is regular, hence λ is Jordan by Fact 2.2(ii) and Claim 2.9(ii).

We have established (i). If ν is nontrivial, then (ii) is done as well. If it is trivial, then λ is nontrivial, so the measure ψ constructed above is a nontrivial normal measure in the open subspace W of X. Therefore Claim 2.9(ii) implies that ${}_X\psi$ is a nontrivial normal measure in X and the proof is complete. $\square$

Theorem 3.3 (Assume $\mathfrak{c} < \mathfrak{r}$). *Let X be a locally compact space such that $\chi(X) < \mathfrak{r}$ and Y a dense-in-itself subspace of X. If Y is locally pre-schismatic, then it is schismatic.*

Proof. As well as in the proof of Lemma 3.2 we may assume that $Y = X$. If X is not schismatic, then Lemma 3.2 yields a normal measure μ in X. Consider its support and the open set $G = \operatorname{int}\operatorname{spt}\mu$. As X is locally pre-schismatic, there is a nonempty open set $U \subseteq G$ that is pre-schismatic. By Fact 2.5(i), μ_G is a nontrivial normal measure in G. We arrived to a contradiction. $\square$

Lemma 3.4. *Let X be a locally compact space and μ a finite, Jordan measure in X. Let $E \subseteq X$.*

(i) *If $d(E) < \mathfrak{m}_{\text{countable}}$, then $\mu(E) = 0$.*
(ii) *If E is measurable and dense-in-itself, and $\hat{\chi}(E) < \mathfrak{m}$, then $\mu(E) = 0$.*

Proof. Both facts are easy corollaries to theorems from [9].

(i) Let $D \subseteq E$ be a dense set such that $|D| < \mathfrak{m}_{\text{countable}}$. Assume that $\mu^\sharp(E) > 0$. Then $\mu^\sharp(D) = \mu(\overline{D}) = \mu(\overline{E}) > 0$, for μ is Jordan, cf. Proposition 2.6(ii). Therefore μ_D is a finite, diffused, nontrivial measure. But by [9, 22H(d) and B1B], each Hausdorff space of cardinality less than $\mathfrak{m}_{\text{countable}}$ is of universal measure zero: a contradiction.

(ii) Put $G = \operatorname{int} E$. If $\mu(E) > 0$, then, by virtue of Claim 2.9(i), μ_G is a nontrivial Jordan measure in a locally compact space G. Since E has no isolated points, neither has G. Hence $\mathfrak{m}_{\text{countable}} > \widehat{\chi}(G) \geqslant \omega_1$, so we infer from Lemmas 3.1 and 3.2(ii) that there is a normal measure ν in G. Consider the support $\operatorname{spt}\nu$ and the measure $\nu_{\operatorname{spt}\nu}$. Let $U \subseteq \operatorname{spt}\nu$ be a nonempty open set with a compact closure. Such a set exists, for G is locally compact and $\operatorname{spt}\nu$ is not nowhere dense. Then $\overline{U}$ is compact and $\mu(\overline{U}) > 0$, so according to Fact 2.5(iii), $\overline{U}$ is ccc. Obviously $\widehat{\chi}(\overline{U}) \leqslant \widehat{\chi}(E) < \mathfrak{m}$. By virtue of [9, 43I(a)], if K is a compact ccc space such that $\widehat{\chi}(K) < \mathfrak{m}$, then K is separable, a contradiction to the just proved part (i). $\square$

Theorem 3.5. *Let X be a locally compact space. If $\widehat{\chi}(X) < \mathfrak{m}$, then each dense-in-itself subspace of X is schismatic.*

Proof. Arguing the same way as in the proof of Lemma 3.2 it is enough to prove the assertion for X itself. If X is not schismatic, by virtue of Lemmas 3.1 and 3.2(i) it is not weakly schismatic. Therefore there is a nontrivial Jordan measure μ in X. Apply Lemma 3.4(ii) to conclude that $\mu(X) = 0$: a contradiction. $\square$

When $\widehat{\chi}(X) < \mathfrak{m}$ is violated enough, then the conclusion of Theorem 3.5 fails.

Proposition 3.6. *Let X be a locally compact space. If $\pi\chi(x, X) \geqslant \mathfrak{c}$ for each $x \in X$, then X contains a dense-in-itself compact subspace that is not pre-schismatic.*

Proof. There is a nonempty open set $U \subseteq X$ the closure of which is compact. Obviously $\pi\chi(x, \overline{U}) \geqslant \pi\chi(x, X) \geqslant \mathfrak{c}$ for all $x \in \overline{U}$, so we may assume that X is compact.

Theorem [14, 3.18] asserts that if X is compact and $\pi\chi(x,X) \geqslant \mathfrak{c}$, then X maps continuously onto $[0,1]^{\mathfrak{c}}$. According to Claim 2.8 there is a dense-in-itself compact subspace of $[0,1]^{\mathfrak{c}}$ that is not pre-schismatic. By [24, Corollary 2.10] and its proof, if $f: X \to Y$ is a perfect mapping onto Y and Y has a dense-in-itself subspace that is not pre-schismatic, then X has such a subspace as well. □

We derive a result about universally measurable spaces. Recall that a completely regular space X is called *universally measurable* if it is measurable, as a subset of its Čech–Stone compactification βX, with respect to each finite measure in βX. The next theorem follows directly from Theorem 3.5 and Lemma 3.4(ii).

Theorem 3.7. *Let X be a completely regular, universally measurable space. If $\widehat{\chi}(X) < \mathfrak{m}$, then each dense-in-itself subspace of X is weakly schismatic.*

Since $\widehat{\chi}X < \omega_2$ means nothing but that X is first countable, Theorem 3.5 and Kunen's example yield the following legible theorem.

Theorem 3.8.

(i) (Assume MA $+ \neg$CH) *If X is a first countable locally compact space, then each dense-in-itself subspace of X is schismatic.*
(ii) (Assume CH) *There is a compact first countable space admitting a strictly positive, finite, diffused residual measure.*

Note that Theorem 3.8 implies that $\widehat{\chi}(X) < \mathfrak{m}$ in Theorems 3.5 and 3.7 cannot be weakened to $\chi(X) < \mathfrak{m}$.

4. Residual measures vs. L-spaces

So the Continuum Hypothesis allows existence of residual measures in first countable compact spaces. Such spaces, however, are then necessarily pathological. Recall that an *L-space* is a regular hereditarily Lindelöff space that is not hereditarily separable and a *Luzin space* is a regular uncountable dense-in-itself space in which every nowhere dense set is countable. Every Luzin space is hereditarily Lindelöf, see [20, 4.3], and thus a nonseparable Luzin space is an L-space.

Proposition 4.1. (Assume CH) *Let X be a locally compact space such that $\chi(X) \leqslant \mathfrak{c}$ and $Y \subseteq X$ a dense-in-itself subspace. If Y is not schismatic, then it contains an L-space.*

Proof. As $\mathfrak{r}$ is weakly inaccessible and $\mathfrak{c} = \omega_1$, it follows that $\chi(Y) < \mathfrak{r}$. Therefore Lemma 3.2(ii) yields a ccc subspace $Z \subseteq Y$ and a strictly positive diffused residual measure μ in Z. By virtue of [13, 2.3], $w(Z) \leqslant \chi(Z)^{\omega}$ and thus $w(Z) \leqslant \mathfrak{c}^{\omega} = \mathfrak{c} = \omega_1$. As Z is a ccc Baire space, the latter property of Z together with CH ensure that there is a family $\{N_\alpha\colon \alpha < \omega_1\}$ of nowhere dense sets such that each nowhere dense subset of Z is covered by some N_α. So one can apply the standard Luzin set construction to find a set L that meets each open subset of Z and misses all but countably many points of each N_α, see [20, 4.3] for details of the construction. Thus L is a Luzin space that is dense in Z. As Z is by virtue of Fact 2.5(ii) nonseparable, it follows that L is not separable as well. Thus L is an L-space. □

Recall that a measure μ is *separable* if $L^1(\mu)$ is, as a metric space, separable. It is easy to check that if μ is a finite separable measure, then $w(\mathcal{S}_\mu) \leqslant \mathfrak{c}$. So Proposition 4.1 yields

Corollary 4.2. (Assume CH) *Let μ be a finite nonatomic separable measure. Then the hyperstonian space $\mathcal{S}_\mu$ of μ contains an L-space.*

Under MA $+ \neg$CH there are no compact or first countable L-spaces, and according to Fact 2.5 a support of a normal measure is always nonseparable and ccc. Thus, any covering property that ensures that ccc subspaces are hereditarily Lindelöff is an enemy of normal measures. Here are some applications of the idea. See the proof *infra* for the definitions of involved covering properties.

Proposition 4.3.

(i) *A regular hereditarily σ-metacompact space that is not pre-schismatic contains an L-space.*

(ii) (Assume MA $+ \neg$CH) *A first countable regular hereditarily σ-metacompact space is pre-schismatic.*

(iii) (Assume MA $+ \neg$CH) *If X is a completely regular universally measurable hereditarily metalindelöff space, then each subspace of X is pre-schismatic.*

Proof. Throughout the proof, F denotes the support of a normal diffused measure. According to Fact 2.5, F is a nonseparable ccc Baire space.

(i) Recall that X is *hereditarily σ-metacompact* if each open family in X has a σ-point-finite open refinement with the same union. By virtue of [23, Theorem 3.8], each point-finite open family in a ccc Baire space is countable. Consequently, each family of open sets in F has a countable open refinement, so F is hereditarily Lindelöff and nonseparable, i.e., F is an L-space.

(ii) If X is not pre-schismatic, then by (i) it contains a first countable L-space. According to [21], under MA $+ \neg$CH there are no first countable L-spaces.

(iii) Recall that X is *hereditarily metalindelöff* if each open family has a point-countable open refinement with the same union.

If μ is a normal measure in X, then ${}_{\beta X}\mu$ is normal in βX and therefore by Claim 2.9(i), $\mu(\mathrm{int}_{\beta X} X) = {}_{\beta X}\mu(\mathrm{int}_{\beta X} X) = {}_{\beta X}\mu(X) = \mu(X) > 0$. Since $\mathrm{int}_{\beta X} X$ is locally compact, it follows that X has a locally compact subspace with a normal measure. Under MA $+ \neg$CH, each point-countable open family in a locally compact ccc space is countable [23, Theorem 5.5], so if F is as above, it is hereditarily Lindelöff and nonseparable. Since the measure is strictly positive on F, *mutatis mutandis* we may assume that F is compact, so F is a compact L-space. According to [13, Corollary to 5.6], under MA + $\neg$CH there are no compact L-spaces. □

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